Durham University

# What Clifford algebra can do for Coxeter groups and root systems

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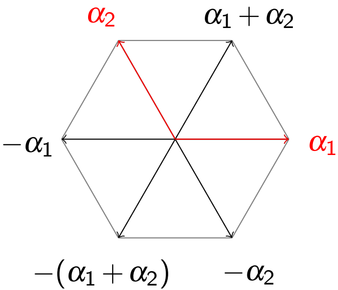
# The general theme: Geometry & Symmetry and their Applications

- Worked on a few different things: HEP – strings, particles and cosmology, pure maths and mathematical biology and Clifford algebras and mathematical physics
- Unifying themes of symmetry and geometry (euclidean, conformal, hyperbolic, spherical)
- Continuous Lie groups, e.g. for modeling cosmological spacetimes (Bianchi models)
- Discrete Coxeter groups and Kac-Moody algebras describe gravitational singularities/hidden symmetries in HEP theory, viruses, fullerenes, &c
- Mathematical frameworks of Coxeter groups and Clifford algebras

# Overview

# Root systems – $A_2$




Root system $\Phi$: set of vectors $\alpha$ such that

$$\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$$

and $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

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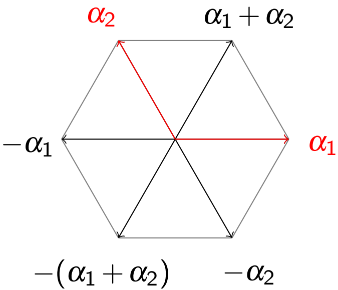


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and $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

Simple roots: express every element of $\Phi$ via a $\mathbb{Z}$-linear combination (with coefficients of the same sign).

# Cartan Matrices

Cartan matrix of $\alpha_i$s is $\boxed{A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij}}$

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$\boxed{\cos^2\theta_{ij} = \frac{1}{4}A_{ij}A_{ji}}$ $\boxed{l_j^2 = \frac{A_{ij}}{A_{ji}}l_i^2}$

$\boxed{A_{ii} = 2}$ $\boxed{A_{ij} \in \mathbb{Z}^{\leq 0}}$ $\boxed{A_{ij} = 0 \Leftrightarrow A_{ji} = 0}$.

$$A_2\colon\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

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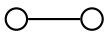
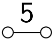
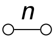
Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label $m$ = angle $\frac{\pi}{m}$.

$A_2$  $H_2$ 

$I_2(n)$ 

# Coxeter groups

A Coxeter group is a group generated by some involutive generators $s_i, s_j \in S$ subject to relations of the form $\boxed{(s_i s_j)^{m_{ij}} = 1}$ with $m_{ij} = m_{ji} \geq 2$ for $i \neq j$.

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The finite Coxeter groups have a geometric representation where the involutions are realised as reflections at hyperplanes through the origin in a Euclidean vector space $\mathscr{E}$. In particular, let $(\cdot|\cdot)$ denote the inner product in $\mathscr{E}$, and $v$, $\alpha \in \mathscr{E}$.

The generator $s_\alpha$ corresponds to the reflection

$$\boxed{s_\alpha : v \rightarrow s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha}$$

at a hyperplane perpendicular to the root vector $\alpha$.

The action of the Coxeter group is to permute these root vectors.

# Coxeter groups vs Lie groups vs Lie algebras vs root systems

- Lie group = group and manifold
- Lie algebra = bilinear, antisymmetric bracket and Jacobi identity
- Lie algebras are infinitesimal version of Lie group = near the identity
- Can be more comprehensive e.g. 2D conformal algebra vs 2D conformal group
- But finite group transformation laws can be easier than linearising
- 'Nice' Lie algebras have triangular decomposition: $\mathscr{N}_- \oplus \mathscr{H} \oplus \mathscr{N}_+$

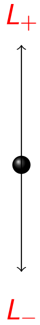
# Coxeter groups vs Lie groups vs Lie algebras vs root systems

- 'Nice' Lie algebras have triangular decomposition: $\mathscr{N}_{-}\oplus\mathscr{H}\oplus\mathscr{N}_{+}$
- $\mathscr{H}$ is the Cartan subalgebra (maximal commuting = quantum numbers)
- Creation and annihilation algebras $\mathscr{N}$ form root lattice
- Symmetry group is called Weyl group and is a crystallographic Coxeter group: $\boxed{A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8}$
- So Coxeter groups in theoretical physics always crystallographic! Neglect $\boxed{I_2(n), H_3, H_4}$.
- Useful Lie algebras are (semi-)simple LA (determinant of Cartan matrix is positive), affine LA (determinant is 0), Kac-Moody algebras, Borcherd's algebras...

# Kac-Moody algebras

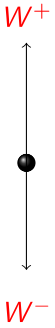
- Kac-Moody algebras $\mathscr{A}$ of rank $N$ are defined by generalised Cartan $(N \times N)$ matrices with $A_{ii} = 2$, $A_{ij} \in Z_-(i \neq j)$ and $A_{ij} \neq 0 \Rightarrow A_{ji} \neq 0$
- $3N$ generators $h_i, e_i, f_i$ satisfy Chevalley-Serre relations $[h_i, h_j] = 0$ $[h_i, e_j] = A_{ij}e_j$, $[h_i, f_j] = -A_{ij}f_j$, $[e_i, f_j] = \delta_{ij}h_i$ $\underbrace{[e_i, [e_i, [\dots, [e_i, e_j]]] \dots]}_{1-A_{ij} \text{ times}} = 0$, $\underbrace{[f_i, [f_i, [\dots, [f_i, f_j]]] \dots]}_{1-A_{ij} \text{ times}} = 0$
- Simple roots $\alpha_i$ are $[h, e_i] = \alpha_i(h)e_i$

# Example – $A_1$, $SU(2)$, Angular Momentum




- Cartan subalgebra = Quantum number: $L_z$
- $\mathcal{N}_+$: raising operator $L_+ = \alpha$
- $\mathcal{N}_-$: lowering operator $L_- = -\alpha$
- ($L^2$ is Casimir/commutes with all algebra elements, is however not actually in the algebra!)

# Example – $A_1$, $SU(2)$, Electroweak




- Cartan subalgebra – Quantum number: $A$
- $\mathscr{N}_+$: raising operator $W^+ = \alpha$
- $\mathscr{N}_-$: lowering operator $W^- = -\alpha$
- (Since SM electroweak is actually $SU(2) \times U(1)$, $U(1)$ gives another field $i$, such that physical $Z^0$ and $\gamma$ are superpositions of $A$ and $i$)
- Also $W^\pm$ now charged and self-interact, unlike QED

# Affine extensions

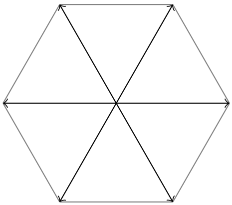
An affine Coxeter group is the extension of a Coxeter group by an affine reflection in a hyperplane not containing the origin $s_{\alpha_0}^{aff}$ whose geometric action is given by

$$\boxed{s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0|v)}{(\alpha_0|\alpha_0)}\alpha_0}$$

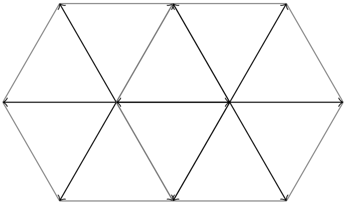
Non-distance preserving: includes the translation generator

$$\boxed{Tv = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v}$$

# Affine extensions – $A_2$

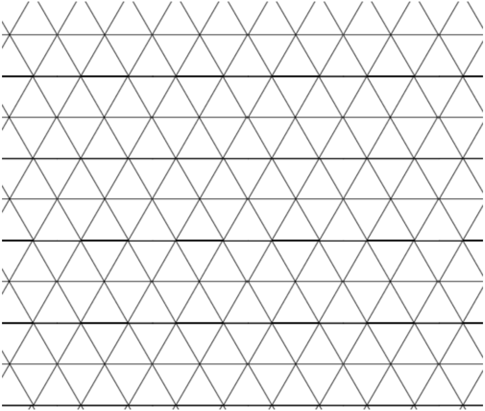


# Affine extensions – $A_2$

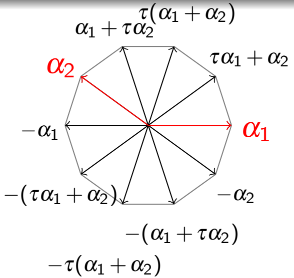


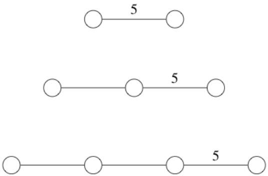
# Affine extensions – $A_2$

Affine extensions of crystallographic Coxeter groups lead to a tessellation of the plane and a lattice.



# Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$




$$A=\begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$

$$A=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

linear combinations now in the extended integer ring

$\boxed{\mathbb{Z}[\tau]=\{a+\tau b | a,b\in\mathbb{Z}\}}$ golden ratio $\boxed{\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}}$

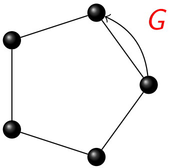
$\boxed{x^2=x+1}$ $\boxed{\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}}$ $\boxed{\tau+\sigma=1, \tau\sigma=-1}$

## What's new?

- In HEP, mostly come from Lie groups, then Lie algebras, then their Weyl groups and root systems
- This only gives the crystallographic Coxeter groups
- Do the non-crystallographic Coxeter groups have something interesting to offer? In particular, affine extensions?
- Interesting connections between the geometries of different dimensions: Relation between crystallographic and non-crystallographic ($E_8$ and $H_4$) and my spinor construction (3 & 4D)
- Both could have interesting consequences for HEP (4D groups and $E_8$ feature heavily) and other applications (viruses, quasicrystals, proteins, fullerenes...)

# Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



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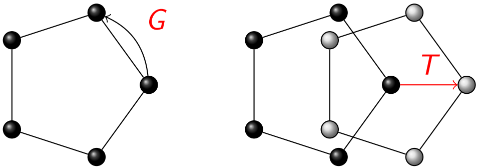

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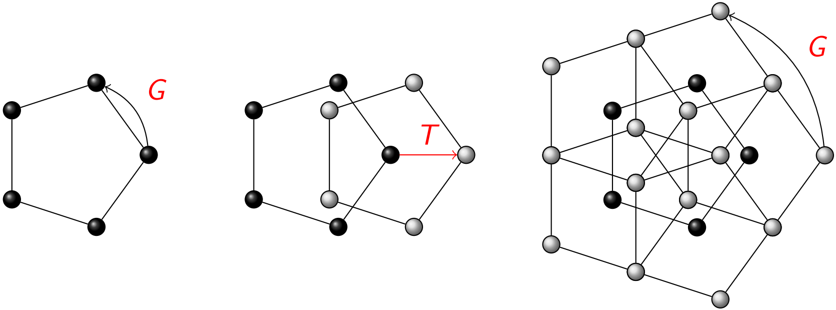
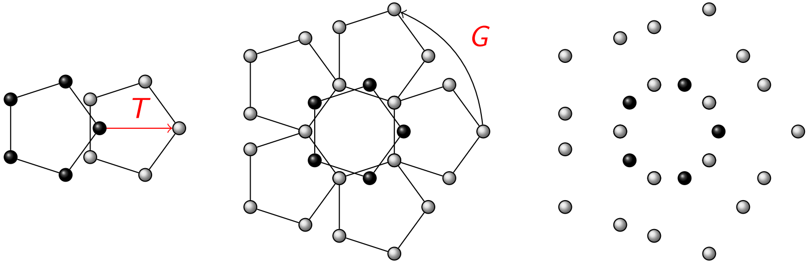



A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to 'coinciding points'.

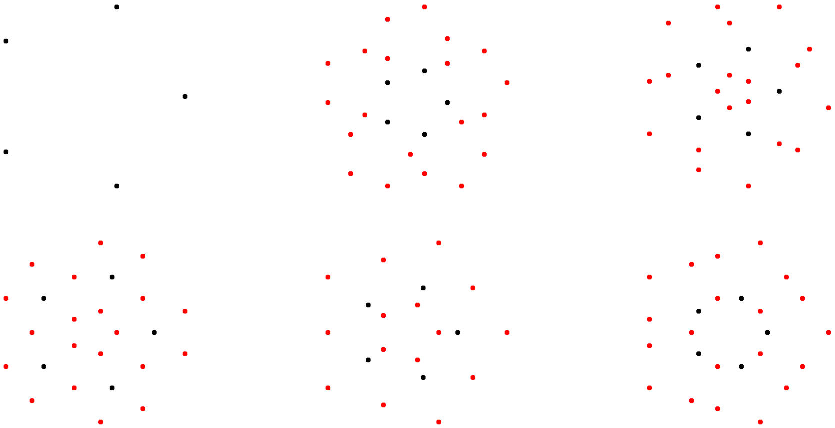
# Affine extensions of non-crystallographic root systems

Translation of length $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (golden ratio)

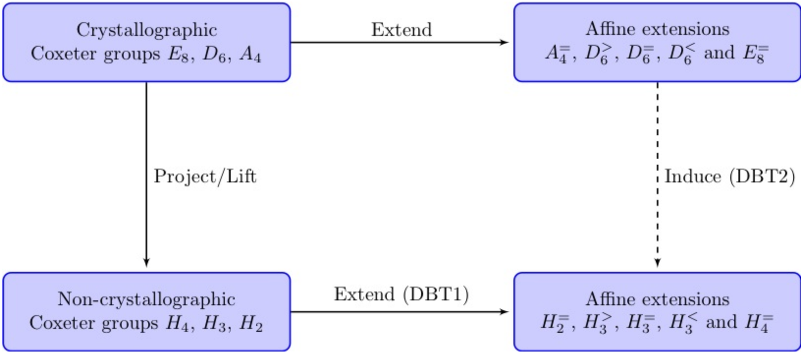



Looks like a virus or carbon onion

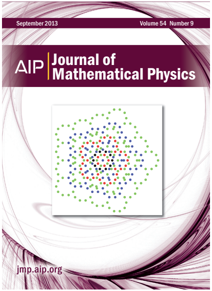
# More Blueprints



# Road Map


Crystallographic
Coxeter groups $E_8$, $D_6$, $A_4$
Extend
Affine extensions
$A_4^=$, $D_6^>$, $D_6^=$, $D_6^<$ and $E_8^=$
Project/Lift
Induce (DBT2)
Non-crystallographic
Coxeter groups $H_4$, $H_3$, $H_2$
Extend (DBT1)
Affine extensions
$H_2^=$, $H_3^>$, $H_3^=$, $H_3^<$ and $H_4^=$

# Applications of affine extensions of non-crystallographic root systems





There are interesting applications to quasicrystals, viruses or carbon onions, but here concentrate on the mathematical aspects

# Basics of Clifford Algebra I

- Form an algebra using the Geometric Product
$\boxed{ab \equiv a \cdot b + a \wedge b}$ for two vectors
- Extend via linearity and associativity to higher grade elements (multivectors)
- For an n-dimensional space generated by n orthogonal unit vectors $e_i$ have $2^n$ elements
- Then $e_i e_j = e_i \wedge e_j = -e_j e_i$ so anticommute (Grassmann variables, exterior algebra)
- Unlike the inner and outer products separately, this product is invertible
- This feeds through to the differential structure of the theory with more powerful Greens functions methods $\nabla^{-1}$

# Basics of Clifford Algebra II

- These are known to have matrix representations over the normed division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ $\Rightarrow$ Classification of Clifford algebras
- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- These have the well-known matrix representations in terms of $\sigma$- and $\gamma$-matrices
- Working with these is not necessarily the most insightful thing to do, so here stress approach to work directly with the algebra
- Naturally have things that square to $-1$, e.g. $\boxed{(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1}$, and non-trivial commutation properties

# Reflections

- Clifford algebra is very efficient at performing reflections
- Consider reflecting the vector $a$ in a hypersurface with unit normal $n$:

$$a' = a_\perp - a_\parallel = a - 2a_\parallel = a - 2(a \cdot n)n$$

- c.f. fundamental Weyl reflection $s_i : v \to s_i(v) = v - 2\frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i$
- But in Clifford algebra have $n \cdot a = \frac{1}{2}(na + an)$ so reassembles into sandwiching

$$\boxed{a' = -nan}$$

- So both Coxeter and Clifford frameworks are ideally suited to describing reflections – first to combine the two

# Reflections and Rotations

- Generate a rotation when compounding two reflections wrt $n$ then $m$ (Cartan-Dieudonné theorem):

$$a'' = mnanm \equiv Ra\tilde{R}$$

  where $R = mn$ is called a rotor and a tilde denotes reversal of the order of the constituent vectors ($R\tilde{R} = 1$)
- Now neat thing is all multivectors transform covariantly e.g.

$$MN \to (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$$

  so transform double-sidedly
- Rotors form a group, the rotor group, which gives a representation of the Spin group $Spin(n)$ – they transform single-sidedly (obvious now it's a double (universal) cover)

# Artin's Theorem and orthogonal transformations

- Artin: every isometry is at most $d$ reflections
- Since have a double cover of reflections ($n$ and $-n$) we have a double cover of $O(p,q)$: $\mathrm{Pin}(p,q)$

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1}$$

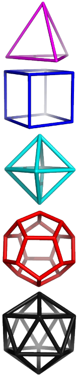
- Pinors/versors = products of vectors $n_1 n_2 \dots n_k$ encode orthogonal transformations via 'sandwiching'
- Cartan-Dieudonné: rotations are an even number of reflections: $\mathrm{Spin}(p,q)$ doubly covers $SO(p,q)$
- The conformal group $C(p,q) \sim SO(p+1,q+1)$ so can use these for translations, inversions etc as well

# Spinor techniques

- Of course there is a matrix representation $\underline{R}$ for the action of a spinor: $\boxed{\underline{R}x = Rx\tilde{R}}$
- This is the usual rotation matrix $\underline{R}$ in $SO(p,q)$
- Having the spin double cover/square root of the rotation matrix can be convenient
- E.g. can get differential equations for spinor $R$ that are easier to solve, then can reconstitute $\underline{R}$ if necessary

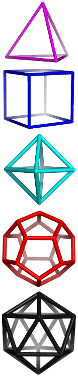
# Overview

## 3D Platonic Solids



- There are 5 Platonic solids
- Tetrahedron (self-dual) ($A_3$)
- Dual pair octahedron and cube ($B_3$)
- Dual pair icoshahedron and dodecahedron ($H_3$)
- Only the octahedron is a root system (actually for ($A_1^3$))

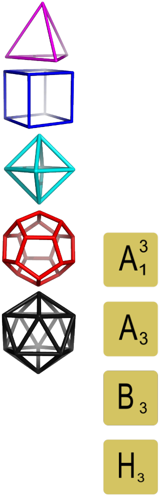
# Clifford and Coxeter: Platonic Solids



| Platonic Solid | Group | root system |
|---|---|---|
| Tetrahedron | $A_3$ | Cuboctahedron |
| | $A_1^3$ | Octahedron |
| Octahedron | $B_3$ | Cuboctahedron |
| Cube | | $+$Octahedron |
| Icosahedron | $H_3$ | Icosidodecahedron |
| Dodecahedron | | |

- Platonic Solids have been known for millennia

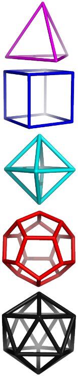
# Clifford and Coxeter: Platonic Solids

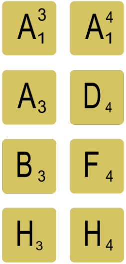



| Platonic Solid | Group | root system |
|---|---|---|
| Tetrahedron | $A_3$<br>$A_1^3$ | Cuboctahedron<br>Octahedron |
| Octahedron<br>Cube | $B_3$ | Cuboctahedron<br>+Octahedron |
| Icosahedron<br>Dodecahedron | $H_3$ | Icosidodecahedron |

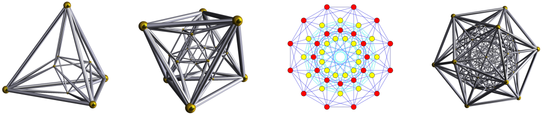
- Platonic Solids have been known for millennia
- Described by Coxeter groups

# Clifford and Coxeter: Platonic Solids






- Platonic Solids have been known for millennia; described by Coxeter groups
- Concatenating reflections gives Clifford spinors (binary polyhedral groups)
- These induce 4D root systems
  $\psi = a_0 + a_i I e_i \Rightarrow \psi\tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- 4D analogues of the Platonic Solids and give rise to 4D Coxeter groups



# 4D 'Platonic Solids'

- In 4D, there are 6 analogues of the Platonic Solids:
- 5-cell (self-dual) ($A_4$)
- 24-cell (self-dual) ($D_4$) – a 24-cell and its dual together are the $F_4$ root system
- Dual pair 16-cell and 8-cell ($B_4$)
- Dual pair 600-cell and 120-cell ($H_4$)
- 24-cell, 16-cell and 600-cell are all root systems, as is the related $F_4$ root system
- 8-cell and 120-cell are dual to a root system, so in 4D out of 6 Platonic Solids only the 5-cell (corresponding to $A_n$ family) is not related to a root system!
- The 4D Platonic solids are not normally thought to be related to the 3D ones except for the boundary cells

# Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

| rank 3 | $\lvert\Phi\rvert$ | $\lvert W\rvert$ | rank 4 | $\lvert\Phi\rvert$ | Symmetry |
|---|---|---|---|---|---|
| $A_3$ | 12 | 24 | $D_4$ 24-cell | 24 | $2\cdot 24^2 = 576$ |
| $B_3$ | 18 | 48 | $F_4$ lattice | 48 | $48^2 = 2304$ |
| $H_3$ | 30 | 120 | $H_4$ 600-cell | 120 | $120^2 = 14400$ |
| $A_1^3$ | 6 | 8 | $A_1^4$ 16-cell | 8 | $3!\cdot 8^2 = 384$ |
| $A_1\oplus A_2$ | 8 | 12 | $A_2\oplus A_2$ prism | 12 | $12^2 = 144$ |
| $A_1\oplus H_2$ | 12 | 20 | $H_2\oplus H_2$ prism | 20 | $20^2 = 400$ |
| $A_1\oplus I_2(n)$ | $n+2$ | $2n$ | $I_2(n)\oplus I_2(n)$ | $2n$ | $(2n)^2$ |

Similar for Grand Antiprism ($H_4$ without $H_2\oplus H_2$) and Snub 24-cell ($2I$ without $2T$).

# Induction Theorem

- Theorem: 3D spinor groups are root systems ($R$ and $-R$ are in a spinor group by construction, and closure under reflections is guaranteed by the closure property of the spinor group)
- Induction Theorem: Every rank-3 root system induces a rank-4 root system.
- Counterexample: not every rank-4 root system is induced in this way
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group

# Induction Theorem

- So induced 4D polytopes are actually root systems.
- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group
- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.
- Only remaining cases in 3D are $A_1 \oplus I_2(n)$, which give $I_2(n) \oplus I_2(n)$

# General Case of Induction

Only remaining case is what happens for $A_1 \oplus I_2(n)$ - this gives a doubling $I_2(n) \oplus I_2(n)$

| rank 3 | rank 4 |
|---|---|
| $A_3$ | $D_4$ |
| $B_3$ | $F_4$ |
| $H_3$ | $H_4$ |
| $A_1^3$ | $A_1^4$ |
| $A_1 \oplus A_2$ | $A_2 \oplus A_2$ |
| $A_1 \oplus H_2$ | $H_2 \oplus H_2$ |
| $A_1 \oplus I_2(n)$ | $I_2(n) \oplus I_2(n)$ |

# Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

| rank 3 | $\|\Phi\|$ | $\|W\|$ | rank 4 | $\|\Phi\|$ | Symmetry |
|---|---|---|---|---|---|
| $A_3$ | 12 | 24 | $D_4$ 24-cell | 24 | $2\cdot 24^2 = 576$ |
| $B_3$ | 18 | 48 | $F_4$ lattice | 48 | $48^2 = 2304$ |
| $H_3$ | 30 | 120 | $H_4$ 600-cell | 120 | $120^2 = 14400$ |
| $A_1^3$ | 6 | 8 | $A_1^4$ 16-cell | 8 | $3!\cdot 8^2 = 384$ |
| $A_1 \oplus A_2$ | 8 | 12 | $A_2 \oplus A_2$ prism | 12 | $12^2 = 144$ |
| $A_1 \oplus H_2$ | 12 | 20 | $H_2 \oplus H_2$ prism | 20 | $20^2 = 400$ |
| $A_1 \oplus I_2(n)$ | $n+2$ | $2n$ | $I_2(n) \oplus I_2(n)$ | $2n$ | $(2n)^2$ |

Similar for Grand Antiprism ($H_4$ without $H_2 \oplus H_2$) and Snub 24-cell ($2I$ without $2T$). Additional factors in the automorphism group come from 3D Dynkin diagram symmetries!

# Some non-Platonic examples of spinorial symmetries

- Grand Antiprism: the 100 vertices achieved by subtracting 20 vertices of $H_2 \oplus H_2$ from the 120 vertices of the $H_4$ root system 600-cell – two separate orbits of $H_2 \oplus H_2$
- This is a semi-regular polytope with automorphism symmetry $\mathrm{Aut}(H_2 \oplus H_2)$ of order $400 = 20^2$
- Think of the $H_2 \oplus H_2$ as coming from the doubling procedure? (Likewise for $\mathrm{Aut}(A_2 \oplus A_2)$ subgroup)
- Snub 24-cell: $2T$ is a subgroup of $2I$ so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group $2T \times 2T$ of order $576 = 24^2$.

# Sub root systems

- The above spinor groups had spinor multiplication as the group operation
- But also closed under twisted conjugation – corresponds to closure under reflections (root system property)
- If we take twisted conjugation as the group operation instead, we can have various subgroups
- These are the remaining 4D root systems e.g. $A_4$ or $B_4$

# What's new?

- Novel connection between geometry of 3D and 4D
- In fact, 3D seems more fundamental – contrary to the usual perspective of 3D subgroups of 4D groups
- Spinorial symmetries
- Clear why spinor group gives a root system and why two factors of the same group reappear in the automorphism group
- Novel spinorial perspective on 4D geometry
- Accidentalness of the spinor construction and exceptional 4D phenomena
- Connection with Arnold's trinities, the McKay correspondence and Monstrous Moonshine

# Recap: Clifford algebra and reflections & rotations

- Clifford algebra is very efficient at performing reflections via sandwiching

$$\boxed{a' = -nan}$$

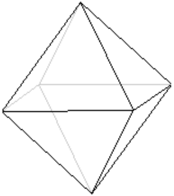
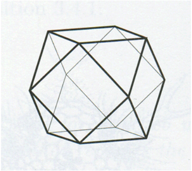
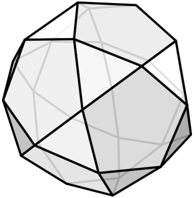
- Generate a rotation when compounding two reflections wrt $n$ then $m$ (Cartan-Dieudonné theorem):

$$\boxed{a'' = mnanm \equiv Ra\tilde{R}}$$

where $R = mn$ is called a rotor and a tilde denotes reversal of the order of the constituent vectors $(R\tilde{R} = 1)$

# From the Coxeter simple roots to the root system

- Take the $A_1 \times A_1 \times A_1$ simple roots $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ $\Rightarrow$ under reflections get $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, the vertices of an octahedron.
- Take the three simple roots of $A_1 \times A_1 \times A_1/A_3/B_3/H_3$. Closure under Clifford reflections generate the whole root system of 6/12/18/30 vertices of an octahedron/cuboctahedron/ cuboctahedron with an octahedron/ icosidodecahedron).

# Spinors from reflections

- These are the 3D Coxeter groups that are symmetry groups of the Platonic Solids (tetrahedron and octahedron are similar but simpler than the icosahedron)
- The 6/12/18/30 reflections in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 rotors.
- E.g. $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ give the 8 permutations of $(\pm 1; 0, 0, 0)$ (scalar and bivector parts, the notation will become clear later).
- The discrete spinor group is isomorphic to the quaternion group $Q$ / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).

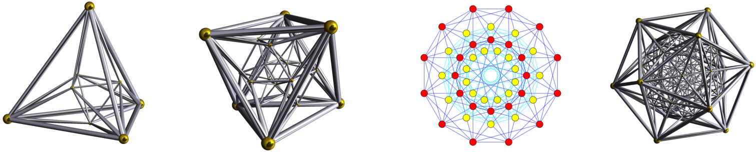
# A unified framework for polyhedral groups

| Group | Discrete subgroup | Action Mechanism |
|---|---|---|
| $SO(3)$ | rotational (chiral) | $x \rightarrow \tilde{R}xR$ |
| $O(3)$ | reflection (full/Coxeter) | $x \rightarrow \pm\tilde{A}xA$ |
| $\mathrm{Spin}(3)$ | binary | $(R_1, R_2) \rightarrow R_1R_2$ |
| $\mathrm{Pin}(3)$ | pinor | $(A_1, A_2) \rightarrow A_1A_2$ |

- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 rotors, which form the binary icosahedral group
- together with the inversion/pseudoscalar $I$ this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group $H_3$ in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in neutrino and flavour physics for family symmetry model building

# Spinors and Polytopes

- The space of $\mathrm{Cl}(3)$-spinors and quaternions have a 4D Euclidean signature: $\psi = a_0 + a_i I e_i \Rightarrow \psi\tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- Can reinterpret spinors in $\mathbb{R}^3$ as vectors in $\mathbb{R}^4$
- Then the spinors constitute the vertices of the 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell
- These are 4D analogues of the Platonic Solids: regular convex 4-polytopes

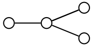


# Spinors, Polytopes and Root systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, $D_4$, $F_4$ and $H_4$
- Exceptional phenomena: $D_4$ (triality, important in string theory), $F_4$ (largest lattice symmetry in 4D), $H_4$ (largest non-crystallographic symmetry)
- Exceptional $D_4$ and $F_4$ arise from series $A_3$ and $B_3$
- In fact, can strengthen this statement on inducing polytopes to statement on inducing root systems

# Root systems in three and four dimensions

The spinors generated from the reflections contained in the respective rank-3 Coxeter group via the geometric product are realisations of the binary polyhedral groups $Q$, $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) rank-4 groups, but not known why, and why the 'mysterious symmetries'.

| rank-3 group | diagram | binary | rank-4 group | diagram |
|---|---|---|---|---|
| $A_1 \times A_1 \times A_1$ | $\circ \quad \circ \quad \circ$ | $Q$ | $A_1 \times A_1 \times A_1 \times A_1$ | $\circ \quad \circ \quad \circ \quad \circ$ |
| $A_3$ | $\circ - \circ - \circ$ | $2T$ | $D_4$ |  |
| $B_3$ | $\circ - \circ \overset{4}{-} \circ$ | $2O$ | $F_4$ | $\circ - \circ \overset{4}{-} \circ - \circ$ |
| $H_3$ | $\circ - \circ \overset{5}{-} \circ$ | $2I$ | $H_4$ | $\circ - \circ - \circ \overset{5}{-} \circ$ |

# Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$
- The Lie Algebras $(E_6, E_7, E_8)$
- The symmetries of the Platonic Solids $(A_3, B_3, H_3)$
- The 4D groups $(D_4, F_4, H_4)$
- New connections via my Clifford spinor construction (see McKay correspondence)

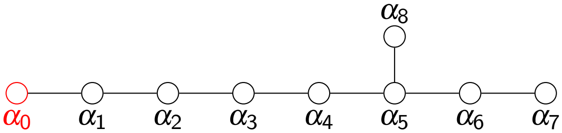
# Platonic Trinities

- Arnold's connection between $(A_3, B_3, H_3)$ and $(D_4, F_4, H_4)$ is very convoluted and involves numerous other trinities at intermediate steps:
- Decomposition of the projective plane into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$
- Notice this miraculously matches the quasihomogeneous weights $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups $(D_4, F_4, H_4)$
- Believe the Clifford connection is more direct

# Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of versors in Clifford
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- All binary are discrete subgroups of $SU(2)$ and all thus have a $2_s$ spinor irrep
- Connection with the McKay correspondence!

# Affine extensions – $E_8^{=}$




$$\boxed{-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8}$$

AKA $E_8^{+}$ and along with $E_8^{++}$ and $E_8^{+++}$ thought to be the underlying symmetry of String and M-theory

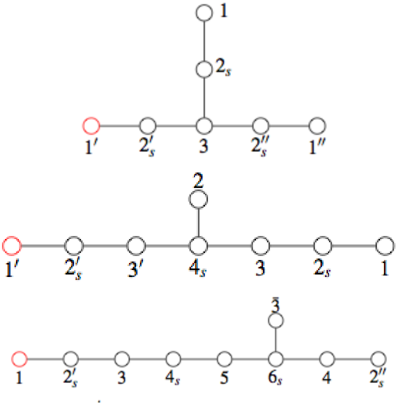
Also interesting from a pure mathematics point of view: $E_8$ lattice, McKay correspondence and Monstrous Moonshine.

# The McKay Correspondence

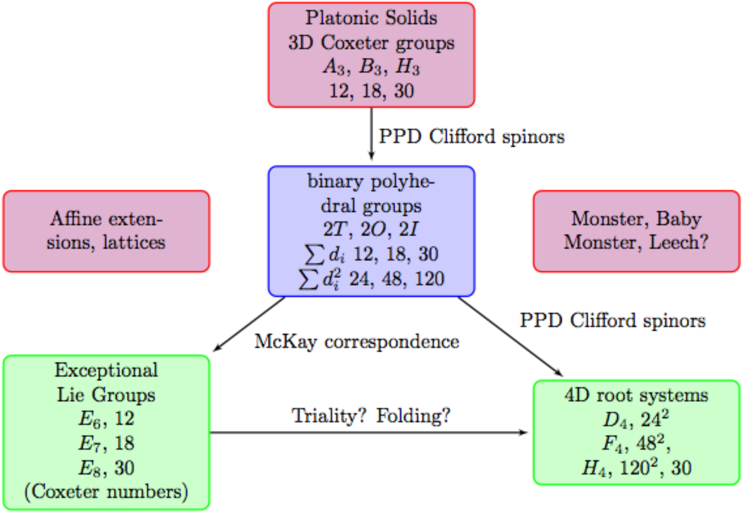
binary polyhedral groups
$2T$, $2O$, $2I$
$\sum d_i$ 12, 18, 30
$\sum d_i^2$ 24, 48, 120

McKay correspondence

Exceptional Lie Groups
$E_6$, 12
$E_7$, 18
$E_8$, 30
(Coxeter numbers)



# The McKay Correspondence



# The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the cyclic and dicyclic groups are in correspondence with $A_n$ and $D_n$, e.g. the quaternion group $Q$ and $D_4^+$. So McKay correspondence not just a trinity but ADE-classification. We also have $I_2(n)$ on top of the trinity $(A_3, B_3, H_3)$

| rank-3 group | diagram | binary | rank-4 group | diagram | Lie algebra | diagram |
|---|---|---|---|---|---|---|
| $A_1 \times A_1 \times A_1$ | | $Q$ | $A_1 \times A_1 \times A_1 \times A_1$ | | $D_4^+$ | |
| $A_3$ | | $2T$ | $D_4$ | | $E_6^+$ | |
| $B_3$ | 4 | $2O$ | $F_4$ | 4 | $E_7^+$ | |
| $H_3$ | 5 | $2I$ | $H_4$ | 5 | $E_8^+$ | |

# 4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- $A_4$ is $SU(5)$ and comes up in Grand Unification
- $D_4$ is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- $B_4$ is $SO(9)$ and is the little group of M-Theory
- $F_4$ is the largest crystallographic symmetry in 4D and $H_4$ is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

# Quaternions and Clifford Algebra

- The unit spinors $\{1; Ie_1; Ie_2; Ie_3\}$ of $\mathrm{Cl}(3)$ are isomorphic to the quaternion algebra $\mathbb{H}$ (up to sign)
- The 3D Hodge dual of a vector is a pure bivector which corresponds to a pure quaternion, and their products are identical (up to sign)

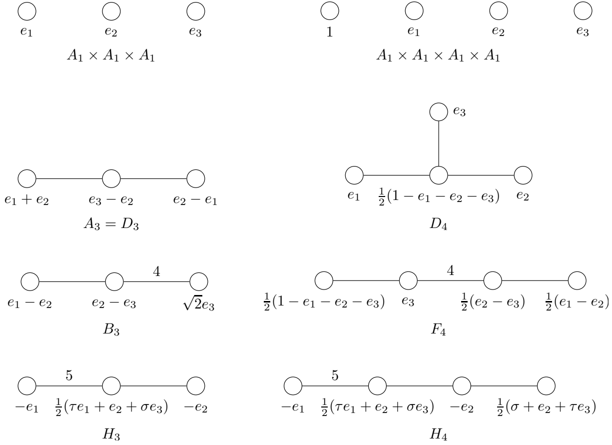
# Discrete Quaternion groups

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are called the Lipschitz units, and form a realisation of the quaternion group in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are called the Hurwitz units, and realise the binary tetrahedral group of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form a group isomorphic to the binary octahedral group of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm\tau, \pm 1, \pm\sigma)$ and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.

# Quaternionic representations of 3D and 4D Coxeter groups

- Groups $E_8$, $D_4$, $F_4$ and $H_4$ have representations in terms of quaternions
- Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. $H_4$ consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm\tau, \pm 1, \pm\sigma)$
- Seen as remarkable that the subset of the 30 pure quaternions is a realisation of $H_3$ (a sub-root system)
- Similarly, $A_3$, $B_3$, $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions
- Will see there is a much simpler geometric explanation

# Quaternionic representations used in the literature



# Demystifying Quaternionic Representations

- 3D: Pure quaternions = Hodge dualised (pseudoscalar) root vectors
- In fact, they are the simple roots of the Coxeter groups
- 4D: Quaternions = disguised spinors – but those of the 3D Coxeter group i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

# Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar $I$
- e.g. does not work for the tetrahedral group $A_3$, but $A_3 \to D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1\alpha_2$ and $R_2 = \alpha_2\alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

# Quaternions vs Clifford versors

- Sandwiching is often seen as particularly nice feature of the quaternions giving rotations
- This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D
- However, the root system construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is self-dual
- Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \to D_4 \oplus D_4$

# References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups
  Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321
- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)
  Advances in Applied Clifford Algebras (2013)
- Nomination for W.K. Clifford Prize (2014)
- Invitation to Arizona State University
- Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction arXiv:1207.7339 (2012)
- Platonic Solids generate their 4-dimensional analogues
  Acta Cryst. A69 (2013)

# Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have invariant polynomials. Their degrees $d$ are important invariants/group characteristics.
- Turns out that actually degrees $d$ are intimately related to so-called exponents $m$ $\boxed{m = d - 1}$.

# Coxeter Elements, Degrees and Exponents

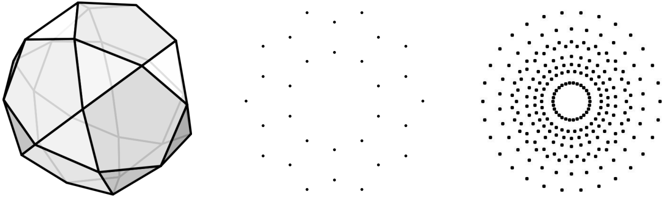
- A Coxeter Element is any combination of all the simple reflections $\boxed{w = s_1 \dots s_n}$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $\boxed{v \rightarrow wv = \pm \tilde{W} v W}$. All such elements are conjugate and thus their order is invariant and called the Coxeter number $h$.
- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where $m$ are called exponents.
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

# Coxeter Elements, Degrees and Exponents

- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where $m$ are called exponents
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, $1$ and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m=d-1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

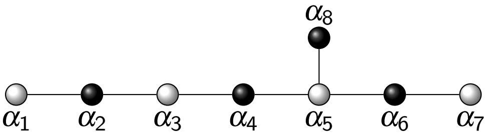
# The Coxeter Plane

- Can show every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane



# The Coxeter Plane

- Obvious from Clifford point of view, that Coxeter element has eigenspaces (eigenblades) rather than just eigenvectors
- In particular, can show every (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of mutually commuting generators



# The Coxeter Plane

- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of orthogonal = mutually commuting generators but anticommuting root vectors $\alpha_w$ and $\alpha_b$ (duals $\omega$)
- Cartan matrices are positive definite, and thus have a Perron-Frobenius (all positive) eigenvector $\lambda_i$.
- Take linear combinations of components of this eigenvector as coefficients of two vectors from the orthogonal sets $v_w = \sum \lambda_w \omega_w$ and $v_b = \sum \lambda_b \omega_b$
- Their outer product/Coxeter plane bivector $\boxed{B_C = v_b \wedge v_w}$ describes an invariant plane where $w$ acts by rotation by $2\pi/h$.

# Clifford Algebra and the Coxeter Plane – 2D case

- For $I_2(n)$ take $\boxed{\alpha_1 = e_1,\ \alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2}$
- So Coxeter versor is just

$$\boxed{W = \alpha_1\alpha_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}e_1e_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}I = -\exp\left(-\frac{\pi I}{n}\right)}$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes rotations in the $e_1e_2$-Coxeter-plane and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $\boxed{w^n = 1}$ via $wv = \tilde{W}vW$.

# Clifford Algebra and the Coxeter Plane – 2D case

- So Coxeter versor is just $\boxed{W = -\exp\left(-\frac{\pi I}{n}\right)}$
- $\boxed{I = e_1 e_2}$ anticommutes with both $e_1$ and $e_2$ such that sandwiching formula becomes $\boxed{v \rightarrow wv = \tilde{W}vW = \tilde{W}^2 v = \exp\left(\pm\frac{2\pi I}{n}\right)v}$ immediately yielding the standard result for the complex eigenvalues in real Clifford algebra without any need for artificial complexification
- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the complex structure
- The Coxeter plane bivector $B_C$ is invariant under the Coxeter versor $\tilde{W}B_C W = \pm B_C$.

# Clifford Algebra and the Coxeter Plane – 3D case

- In 3D, $A_3$, $B_3$, $H_3$ have $\{1,2,3\}$, $\{1,3,5\}$ and $\{1,5,9\}$
- Coxeter element is product of a spinor in the Coxeter plane with the same complex structure as before, and a reflection perpendicular to the plane
- So in 3D still completely determined by the plane
- $1$ and $h-1$ are rotations in Coxeter plane
- $h/2$ is the reflection (for $v$ in the normal direction)

$$\boxed{wv = \tilde{W}^2 = \exp(\pm\frac{2\pi I}{h}\frac{h}{2}) = \exp(\pm\pi I)v = -v}$$

# Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise $W$ into orthogonal (commuting/anticommuting) components $\boxed{W = \alpha_1 \dots \alpha_n = W_1 \dots W_n}$ with $\boxed{W_i = \exp(\pi m_i I_i / h)}$
- Here, $I_i$ is a bivector describing a plane with $\boxed{I_i^2 = -1}$
- For $v$ orthogonal to the plane descrbed by $I_i$ we have $\boxed{v \to \tilde{W}_i v W_i = \tilde{W}_i W_i v = v}$ so cancels out
- For $v$ in the plane we have $\boxed{v \to \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v}$
- Thus if we decompose $W$ into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

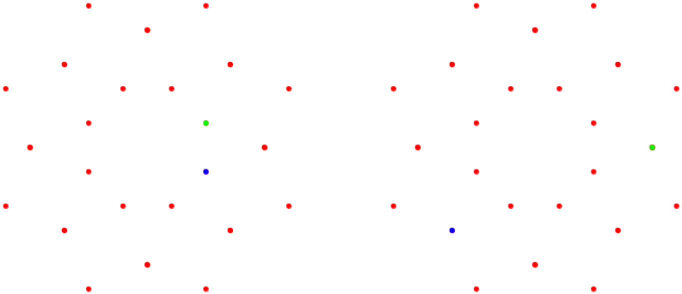
# Clifford algebra: no need for complexification

- For $v$ in the plane we have
$$\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i/h) v}$$
- So complex eigenvalue equation arises geometrically without any need for complexification
- Different complex structures immediately give different eigenplanes
- Eigenvalues/angles/exponents given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. $B_4$ has exponents $1, 3, 5, 7$ and $W = \exp\left(\frac{\pi}{8} I_1\right) \exp\left(\frac{3\pi}{8} I_2\right)$
- Here we have been looking for orthogonal eigenspaces, so innocuous – different complex structures commute
- But not in general – naive complexification can be misleading

# Clifford Algebra and the Coxeter Plane – 4D case

- E.g. $B_4$ has exponents $1, 3, 5, 7$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)}$$



# Clifford Algebra and the Coxeter Plane – 4D case

| rank 4 | exponents | W-factorisation |
|---|---|---|
| $A_4$ | $1, 2, 3, 4$ | $W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} IB_C\right)$ |
| $B_4$ | $1, 3, 5, 7$ | $W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} IB_C\right)$ |
| $D_4$ | $1, 3, 3, 5$ | $W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} IB_C\right)$ |
| $F_4$ | $1, 5, 7, 11$ | $W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} IB_C\right)$ |
| $H_4$ | $1, 11, 19, 29$ | $W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} IB_C\right)$ |

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

# Clifford Algebra and the Coxeter Plane – $D_6$

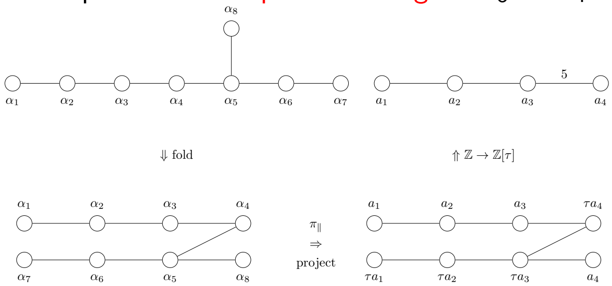
- For $D_6$ one has exponents $\boxed{1, 3, 5, 5, 7, 9}$
- Coxeter versor decomposes into orthogonal bits as

$$\boxed{W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right) \exp\left(\frac{3\pi}{10}B_2\right)}$$

- Now bivector exponentials correspond to rotations in orthogonal planes
- Vector factors correspond to reflections
- For odd $n$, there is always one such vector factor in $D_n$, and for even $n$ there are two

# Projection and Diagram Foldings

Compare with the 'partial folding' of $E_8$ to $H_4$




$$\boxed{s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \ s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \ s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \ s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4}$$

# Imaginary differences – different imaginaries

So what has been gained by this Clifford view?

- There are different entities that serve as unit imaginaries
- They have a geometric interpretation as an eigenplane of the Coxeter element
- These don't need to commute with everything like $i$ (though they do here – at least anticommute. But that is because we looked for orthogonal decompositions)
- But see that in general naive complexification can be a dangerous thing to do – unnecessary, issues of commutativity, confusing different imaginaries etc

# Conformal geometry and Clifford algebra

- The conformal group $C(p,q) \sim SO(p+1,q+1)$
- So can use versor representation of conformal transformations in Clifford algebra (reflections, translations, inversions ...)
- Treat all of them multiplicatively in terms of versors and use sandwiching $\boxed{Ax\tilde{A}}$
- E.g. can generate a whole root lattice multiplicatively with compact reflection part and translations

# Conformal Clifford Algebra

- The conformal group $C(n,p)$ is homomorphic to $Spin(n+1,p+1)$
- Go to $e_1, e_2, e_3, e, \bar{e}$, with $e_i^2 = 1, e^2 = 1, \bar{e}^2 = -1$
- Define two null vectors $n \equiv e + \bar{e},\ \bar{n} \equiv e - \bar{e}$
- Can embed the 3D vector $x = x^\mu e_\mu = xe_1 + ye_2 + ze_3$ as a null vector in 5D ($\hat{X} \cdot n = -1$)

$$F(x) \equiv \hat{X} = \frac{1}{2\lambda^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- Essentially linear action of $SO(n+1,p+1)$ in embedding space induces a non-linear realisation of the conformal group on the projective light cone (Dirac/Hestenes/Lasenby)
- So neat thing is that conformal transformations are now done by rotors (except inversion which is a reflection) – distances are given by inner products

# Operations in Conformal Geometric Algebra

- Amsterdam protocol: $e = e_+$, $\bar{e} = e_-$, $n = n_\infty$ and $\bar{n} = n_0$.
- Reflections $\boxed{y' = -xyx}$ since $e$ and $\bar{e}$ $\Rightarrow$ $n$ and $\bar{n}$ are orthogonal to $x$ $\Rightarrow$ anticommute $-xnx = n$ and $-x\bar{n}x = \bar{n}$:

$$\boxed{-xF(y)x = F(y') = F(-xyx)}$$

- Rotations $\boxed{y' = Ry\tilde{R}}$ from reflections via Cartan-Dieudonné

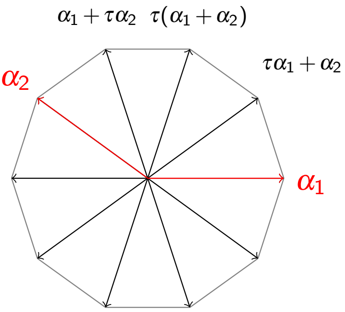
$$\boxed{RF(y)\tilde{R} = F(y') = F(Ry\tilde{R})}$$

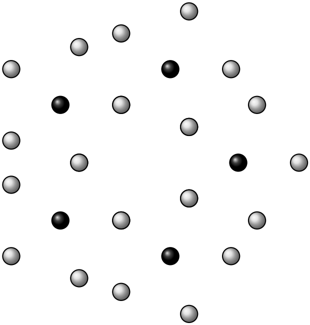
- Translations $\boxed{y' = y + a}$ rotor $\boxed{T_a = \exp\left(\frac{na}{2\lambda}\right) = 1 + \frac{na}{2\lambda}}$

$$\boxed{T_aF(y)\tilde{T}_a = F(y') = F(y + a)}$$

# Proof of Principle

Construction of root systems and quasicrystalline point arrays carries through, e.g. here for $H_2$ and a pentagon with translation $1/\tau$



# Benefits of this approach

- Conceptual Unification of Rotations and Translations via rotors
- Construct root system from the simple roots as before, and likewise for quasicrystalline point arrays
- Increased numerical stability (not really an issue here) due to projective representation

# A new set of Bianchi IX Killing Vectors

- Used Conformal Clifford algebra setup to treat conformal group $C(1,3)$ as $SO(2,4)$
- Stabiliser subgroup of a certain vector gives the de Sitter group (Killing vectors)
- Using a certain projection broke this down to two commuting $SU(2) \times SU(2)$
- This is a new set of Bianchi IX Killing vectors with nice symmetry properties

# A unified framework for polyhedral groups

| Group | Discrete subgroup | Action Mechanism |
|---|---|---|
| $SO(3)$ | rotational (chiral) | $x \rightarrow \tilde{R}xR$ |
| $O(3)$ | reflection (full/Coxeter) | $x \rightarrow \pm\tilde{A}xA$ |
| $\mathrm{Spin}(3)$ | binary | $(R_1, R_2) \rightarrow R_1R_2$ |
| $\mathrm{Pin}(3)$ | pinor | $(A_1, A_2) \rightarrow A_1A_2$ |

- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 rotors, which form the binary icosahedral group
- together with the inversion/pseudoscalar $I$ this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group $H_3$ in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in neutrino and flavour physics for family symmetry model building

# Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of versors in Clifford
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2'_s, 3, \overline{3}, 4, 4_s, 5, 6_s$
- All binary are discrete subgroups of $SU(2)$ and all thus have a $2_s$ spinor irrep
- See McKay correspondence
- Interesting to look at spinors/binary groups in their own right – see Induction Theorem

# Some Group Theory: chiral, full, binary, pin

- Full (Coxeter) is just two copies of this (24/48/120 i.e. same order as binary since both $\mathrm{Spin}(3)$ and $O(3)$ are double covers of $SO(3)$)
- Pin group is just $1+I$ of this for $B_3$ and $H_3$, which contain the inversion $I$
- but not for $A_3$! (which doesn't – c.f. quaternionic reps)
- Instead $\mathrm{Pin}(A_3)$ has the same conjugacy classes as $\mathrm{Spin}(B_3)$

# Conjugacy Classes: Quaternion group $Q$

- Five conjugacy classes: $\{1\}$, $\{-1\}$, $\{\pm e_1 e_2\}$, $\{\pm e_2 e_3\}$, $\{\pm e_3 e_1\}$
- Different conjugacy classes correspond to different geometric subspaces in the Clifford algebra
- Bit trivial for the quaternion group, but extends to arbitrary dimension

# Conjugacy Classes: Binary octahedral group $2O$

- Eight conjugacy classes: $\{1\}$, $\{-1\}$, $\underline{6}$: bivectors $\{\pm e_1e_2, \pm e_2e_3, \pm e_3e_1\}$; $\underline{6}'$: bivector exponentials $\exp\underline{6}$; $\underline{6}''$: $\exp^{-}\underline{6}$; $\underline{8}$: spinors $\{1\pm e_1e_2 \pm e_2e_3 \pm e_3e_1, \dots\}$; $\underline{8}'$: $-\underline{8}$; $\underline{12}$: bivectors $\{e_1(e_2+e_3), \dots\}$;
- Turns out most of these are the same as for $\mathrm{Pin}(A_3)$, and the remaining ones can be mapped to each other
- Though in $\mathrm{Pin}(A_3)$ also have odd grade elements, so some of the conjugacy classes are vector+trivector etc, i.e. different geometric interpretation

# Character tables: Quaternion group $Q$ (from $A_1^3 \to A_1^4$)

| $Q$ | $1$ | $-1$ | $\pm i$ | $\pm j$ | $\pm k$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1'$ | $1$ | $1$ | $1$ | $-1$ | $-1$ |
| $1''$ | $1$ | $1$ | $-1$ | $1$ | $-1$ |
| $1'''$ | $1$ | $1$ | $-1$ | $-1$ | $1$ |
| $2$ | $2$ | $-2$ | $0$ | $0$ | $0$ |
| $4$ | $4$ | $-4$ | $0$ | $0$ | $0$ |

Latter is of quaternionic type – somehow seen as particularly noteworthy

# Character tables: binary octahedral group $2O$ (from $B_3 \to F_4$)

| $2O$ | 1 | 1 | 6 | 8 | 8 | 6 | 6 | 12 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $1'$ | 1 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| 2 | 2 | 2 | 2 | $-1$ | $-1$ | 0 | 0 | 0 |
| 3 | 3 | 3 | $-1$ | 0 | 0 | 1 | 1 | $-1$ |
| $3'$ | 3 | 3 | $-1$ | 0 | 0 | $-1$ | $-1$ | 1 |
| 4 | 4 | $-4$ | 0 | 2 | $-2$ | $2\sqrt{2}$ | $-2\sqrt{2}$ | 0 |
| $4'$ | 4 | $-4$ | 0 | 2 | $-2$ | $-2\sqrt{2}$ | $2\sqrt{2}$ | 0 |
| 8 | 8 | $-8$ | 0 | $-2$ | 2 | 0 | 0 | 0 |

Again some of quaternionic type

# Representations

- This Clifford multivector construction of the polyhedral groups is a faithful realisation/representation, i.e. is essentially the same as the abstract group
- But can define several different representations from these versor groups (may or may not be irreducible ones)
- Representations: matrices $D(R)$ such that

$$\boxed{D(R_1R_2) = D(R_1)D(R_2)}$$

# Representations

- Representations: matrices $D(R)$ such that

$$\boxed{D(R_1R_2) = D(R_1)D(R_2)}$$

- Trivial representation: $D(R) = R1\tilde{R} = 1$
- Rotation representation: for nD vector $x = \sum a_i e_i$:
  $\boxed{D(R)\underline{x} = Rx\tilde{R}}$ usual $SO(n)$ $n \times n$-matrix
- Full representation: for nD vector $x = \sum a_i e_i$: $\boxed{D(A)\underline{x} = Ax\tilde{A}}$
  usual $O(n)$ $n \times n$-matrix
- Spinor representation: for nD spinor $y$ ($2^{n-1}$ components):
  $\boxed{D(R)\underline{y} = Ry}$ a $2^{n-1} \times 2^{n-1}$-matrix
- Versor representation: for nD versor $z$ ($2^n$ components):
  $\boxed{D(A)\underline{z} = Az}$ a $2^n \times 2^n$-matrix

# Character tables and Clifford reps: quaternion group $Q$

The spinor representation $\boxed{D(R)\underline{y} = R\underline{y}}$ of the quaternion group $Q$ gives the representation of quaternionic type. (The trace of $D(R)$ is the character.)

Again just seen to be a consequence of the accidental isomorphism between 3D spinors and quaternions.

| $Q$ | $1$ | $-1$ | $\pm i$ | $\pm j$ | $\pm k$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1'$ | $1$ | $1$ | $1$ | $-1$ | $-1$ |
| $1''$ | $1$ | $1$ | $-1$ | $1$ | $-1$ |
| $1'''$ | $1$ | $1$ | $-1$ | $-1$ | $1$ |
| $2$ | $2$ | $-2$ | $0$ | $0$ | $0$ |
| $4$ | $4$ | $-4$ | $0$ | $0$ | $0$ |

# Character tables and Clifford reps: binary octahedral group $2O$

The spinor representation $\boxed{D(R)\underline{y} = R\underline{y}}$ of the quaternion group $2O$ gives the irrep of quaternionic type.

The rotation representation $\boxed{D(R)\underline{x} = R\underline{x}\tilde{R}}$ gives 3 irrep.

| $2O$ | 1 | 1 | 6 | 8 | 8 | 6 | 6 | 12 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $1'$ | 1 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| 2 | 2 | 2 | 2 | $-1$ | $-1$ | 0 | 0 | 0 |
| 3 | 3 | 3 | $-1$ | 0 | 0 | 1 | 1 | $-1$ |
| $3'$ | 3 | 3 | $-1$ | 0 | 0 | $-1$ | $-1$ | 1 |
| 4 | 4 | $-4$ | 0 | 2 | $-2$ | $2\sqrt{2}$ | $-2\sqrt{2}$ | 0 |
| $4'$ | 4 | $-4$ | 0 | 2 | $-2$ | $-2\sqrt{2}$ | $2\sqrt{2}$ | 0 |
| 8 | 8 | $-8$ | 0 | $-2$ | 2 | 0 | 0 | 0 |

# Clifford: groups and representations summary

- Clifford algebra provides a unified framework for chiral/full/binary/pin polyhedral groups all in the same representation/realisation space/algebra (c.f. usual $SO(3)$ vs $SU(2)$ representations)
- Structure of the algebra $\Rightarrow$ different conjugacy classes are different kinds of objects in the algebra and are kept separate
- Several representations follow, in particular have geometric insight into complex and quaternionic representations

# Clifford summary

- Interesting induction theorem linking geometry of 3D and 4D
- Geometric complex structures and non-trivial commutativity properties
- Simple versor representation of orthogonal transformations
- Conformal geometry
- Some interesting new results on group and representation theory
- Lie algebras can be constructed in Clifford algebra as bivector algebras, and Lie groups as spin groups (work with Phoenix)

# Overview

# Monstrous Moonshine

- Mysterious connection between two very different areas of Mathematics
- Modular forms (functions that live on a torus with complex structure $\tau$): Fourier expansion coefficients wrt $(q = e^{2\pi i\tau})$
- Finite simple groups: dimensions of irreducible representations
- Monstrous Moonshine: The largest sporadic group, the Monster $M$ ($\{1, 196883, 21296876, \ldots\}$) and the Klein $j(\tau)$ modular function
- $\boxed{j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots}$
- $196884 = 196883 + 1$, $21493760 = 21296876 + 196883 + 1$, $\ldots$

# Mathieu Moonshine

- Similar Moonshine phenomenon
- Modular form: elliptic genus of an $\mathcal{N}=4$ SCFT compactified on a K3-surface
- Finite simple group: Mathieu $M_{24}$ $(\{45, 231, 770, 2277, 5796\dots\})$
- Elliptic genus is $\boxed{E_{K3}(\tau,z) = -2Ch(0;\tau,z) + 20Ch(1/2;\tau,z) + e(q)Ch(\tau,z)}$ where all the coefficients in the $q$-series $\boxed{e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots}$ are twice the dimension of some $M_{24}$ irrep

# Clifford and Moonshine

- Looking at Wess-Zumino-Witten models, i.e. strings propagating on a Lie group manifold
- Condition of extended supersymmetry ultimately hinges on classification of Clifford algebras – deep connection
- Connections with binary polyhedral groups, Monstrous Moonshine, McKay correspondence, lattices, affine extensions, Lie groups/algebras etc
- Elliptic genus is constant on (connected components of) the moduli space: better understanding as a topological feature as the index of a Dirac operator?

Thank you!

# Back to the roots

- Unifying principles of geometry and symmetry
- Discrete groups (finite simple groups, polyhedral groups) and continuous groups (string compactifications, non-linear $\sigma$-models)
- Exceptional phenomena: $E_8$, $H_4/F_4/D_4$ (spinorial), McKay correspondence, Monster $M$, Mathieu $M_{24}$ ...
- Applications: from mundane (viruses, fullerenes, quasicrystals) to exotic (HEP, Moonshine) – same mathematical tools: Coxeter, Clifford, affine extensions etc

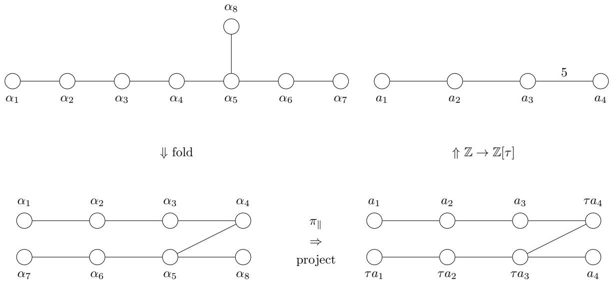
# Mathematical Aspects: Better understanding of geometrical structures

- Affine extensions and translations – lattices, quasicrystals and projections
- Coxeter groups and Kac-Moody theory
- Euclidean, spherical, hyperbolic and conformal geometry
- Clifford and spinorial geometry
- Group theory, Lie groups and algebras (with Phoenix)
- Mathieu Moonshine, McKay correspondence
- Implications for the real world

# HEP

- Gravitational and cosmological singularities – hidden symmetries
- New uses for non-crystallographic groups
- Topological defects
- Integrable systems
- Family symmetries (flavour/neutrino physics)
- 4D groups: $A_4$, $B_4$, $D_4$ – $F_4$, $H_4$?
- Spinor geometry
- $E_8$ features prominently – relation with $H_4$

# Projection and Diagram Foldings




$$\boxed{s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \; s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \; s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \; s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4}$$

$E_8$ has two $H_4$-invariant subspaces – blockdiagonal form

$D_6$ has two $H_3$-invariant subspaces

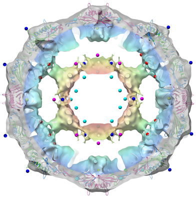
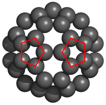
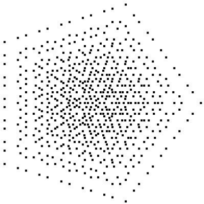
$A_4$ has two $H_2$-invariant subspaces

# 4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- $A_4$ is $SU(5)$ and comes up in Grand Unification
- $D_4$ is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- $B_4$ is $SO(9)$ and is the little group of M-Theory
- $F_4$ is the largest crystallographic symmetry in 4D and $H_4$ is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

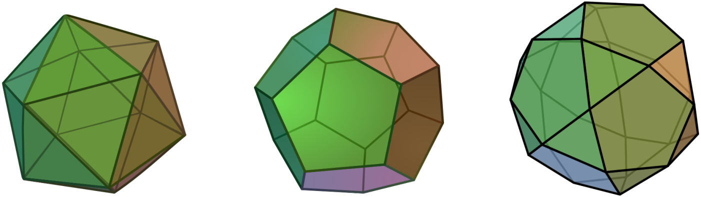
# Motivation: Viruses

- Geometry of polyhedra described by Coxeter groups
- Viruses have to be ‘economical’ with their genes
- Encode structure modulo symmetry
- Largest discrete symmetry of space is the icosahedral group
- Many other ‘maximally symmetric’ objects in nature are also icosahedral: Fullerenes & Quasicrystals
- But: viruses are not just polyhedral – they have radial structure. Affine extensions give translations

# Extend icosahedral group with distinguished translations

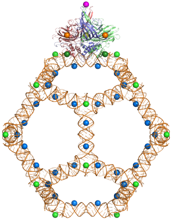
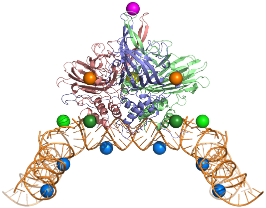
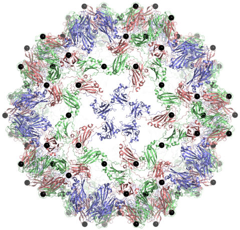
- Radial layers are simultaneously constrained by affine symmetry
- Works very well in practice: finite library of blueprints
- Select blueprint from the outer shape (capsid)
- Can predict inner structure (nucleic acid distribution) of the virus from the point array



Affine extensions of the icosahedral group (giving translations) and their classification.

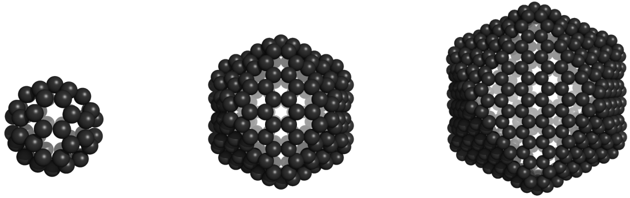
# Use in Mathematical Virology

- Suffice to say point arrays work very exceedingly well in practice. Two papers on the mathematical (Coxeter) aspects.
- Implemented computational problem in Clifford – some very interesting mathematics comes out as well (see later).

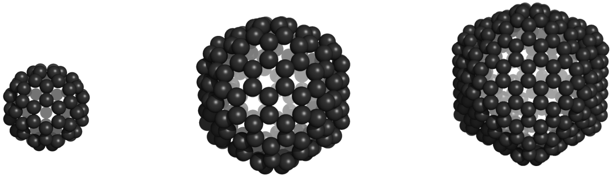
# Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{60} - C_{240} - C_{540}$)



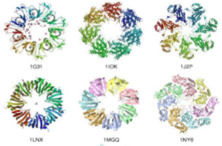
# Extension to fullerenes: carbon onions

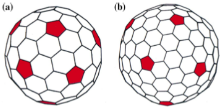
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{80} - C_{180} - C_{320}$)

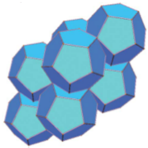


# Applied areas (EPSRC proposal)

- Viruses: Extend to large viruses; interesting results on higher-order translations
- Proteins: Extend affine symmetry to $2D$ and apply to (chiral) proteins
- Fullerenes: Extend to larger fullerenes, in particular chiral carbon onions
- Packings: Novel analytical and numerical approaches to packings of polyhedral solids; with Colapinto (Santa Barbara), Twarock (York) and Thorpe (Phoenix)



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